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A NEW REPRESENTATION OF THE TWO-DIMENSIONAL EQUATIONS OF THE DYNAMICS OF AN INCOMPRESSIBLE FLUID†

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A new representation is obtained for the equations describing the dynamics of an incompressible fluid in a rotating plane and the fundamental properties of this representation are considered. New exact solutions, which describe steady flows of an ideal fluid and, in particular, exact steady-state solutions for waves close to the critical layer are constructed using it.

1. The equations of motion of an incompressible fluid in a rotating plane when there is external friction and viscosity can be written in the form

$$\begin{aligned} &u_t + uu_x + \upsilon u_y - f_0\upsilon + p_x - \kappa(t)u - \nu\Delta u = 0 \\ &\upsilon_t + u\upsilon_x + \upsilon\upsilon_y + f_0 u + p_y - \kappa(t)\upsilon - \nu\Delta\upsilon = 0 \\ &u_x + \upsilon_y = 0(\Delta = \partial^2 / \partial^2 x + \partial^2 / \partial^2 y) \end{aligned} \tag{1.1}$$

Here, f_0 = const is the Coriolis parameter, $\kappa = \kappa(t)$ is the coefficient of external friction, $p = p(x, y, t)$ is the pressure divided by the density ρ = const, u, υ are the velocity components in a Cartesian system of coordinates (x, y) and ν is the coefficient of kinematic viscosity. On introducing the stream function ψ: $u = \partial\psi/\partial y$, $\upsilon = -\partial\psi/\partial x$, the pressure p is traditionally eliminated from the first two equations of (1.1) by cross differentiation, which gives the single equation in the stream function

$$\frac{\partial}{\partial t}\Delta\psi + \frac{\partial}{\partial x}\psi\frac{\partial}{\partial y}\Delta\psi - \frac{\partial}{\partial y}\psi\frac{\partial}{\partial x}\Delta\psi = \nu\Delta\Delta\psi + \kappa\Delta\psi \tag{1.2}$$

In the case of steady inviscid flows, this equation can be written in the form

$$\Delta\psi = F(\psi) \tag{1.3}$$

where $F(\psi)$ is a certain function of arbitrary form which is determined by the boundary conditions. As a rule, Eq. (1.3) is the starting point in searches for the exact solutions of the equations of the dynamics of an ideal fluid [1, 4, 5]. There has been increasing interest in these recently in view of the success of the inverse scattering problem method (ISPM) in the theory of integrable non-linear partial differential equations [3].

Below, we present an alternative approach to the investigation of Eqs (1.1) which is based on their successive analysis as a combination of three generalized differential laws of conservation of momentum flow and mass. As a result, this approach yields a new general representation for Eqs (1.2) and (1.3) which leads to certain new exact relations and to exact solutions in the theory of incompressible fluid flows.

Using the stream function ψ, Eqs (1.1) can be rewritten in the form of a differential law for the conservation of momentum flow per unit area

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$$\frac{\partial}{\partial y}(\psi_t - \nu\Delta\psi - \kappa\psi + uv) + \frac{\partial}{\partial x}(u^2 + f_0\psi + p) = 0$$

$$\frac{\partial}{\partial x}(-\psi_t + \nu\Delta\psi + \kappa\psi + uv) + \frac{\partial}{\partial y}(v^2 + f_0\psi + p) = 0$$

which is equivalent to the following relations

$$\begin{aligned} &u^2 + f_0\psi + p = -h_y, v^2 + f_0\psi + p = -g_x \\ &\psi_t - \nu\Delta\psi - \kappa\psi + uv = h_x, \quad -\psi_t + \nu\Delta\psi + \kappa\psi + uv = g_y \end{aligned} \tag{1.4}$$

where $h = h(x, y, t)$ and $g = g(x, y, t)$ are certain auxiliary functions. Using the substitutions

$$h = \frac{\partial}{\partial y}\ln\phi + h_0(x,y,t), \quad g = \frac{\partial}{\partial x}\ln\phi + g_0(x,y,t)$$

$$\psi = \ln\phi(x,y,z), \quad E(x,y,z) = 2(p + f_0\psi)$$

and some reduction Eqs (1.4) can be written in the form

$$\begin{aligned} &E = -\phi^{-1}\Delta\phi + (h_{0y} + g_{0x}) \\ &\phi_t - (\nu\Delta\ln\phi - \kappa\ln\phi)\phi = (h_{0x} - g_{0y})\phi/2 \\ &\phi_{xx} - \phi_{yy} = (h_{0y} - g_{0x})\phi, \quad -2\phi_{xy} = (h_{0x} + g_{0y})\phi \end{aligned} \tag{1.5}$$

The last three equations can be represented in a more compact complex form

$$\begin{aligned} &\phi_t - (\nu\Delta\ln\phi - \kappa\ln\phi)\phi = i(\overline{R}_z - R_{\bar z})\phi/2 \\ &2\phi_{zz} = R_z\phi, \quad 2\phi_{\bar z\bar z} = \overline{R}_{\bar z}\phi \quad R(z,\bar z,t) = ih_0 - g_0 \\ &z = x + iy, \quad \partial_z = \frac{1}{2}(\partial_x - i\partial_x), \quad \Delta = 4\partial_z\partial_{\bar z} \end{aligned} \tag{1.6}$$

The substitution $R(z, \bar z, t) = \partial Q(z, \bar z, t)/\partial z$ where $Q(z, \bar z, t) = A(z, \bar z, t) + iB(z, \bar z, t)$ and the functions $A(z, \bar z, t)$ and $B(z, \bar z, t)$ are real, reduces system (1.6) to the form

$$\phi_t - (\nu\Delta\ln\phi - \kappa\ln\phi)\phi = \phi\Delta B/4 \tag{1.7}$$

$$2\phi_{zz} = Q_{zz}\phi, \quad 2\phi_{\bar z\bar z} = \overline{Q}_{\bar z\bar z}\phi$$

while the first equation of system (1.4) for E takes the form

$$E = -\phi^{-1}\Delta\phi + \Delta A/2 \tag{1.8}$$

It is proved in this manner that the initial system of Eqs (1.1) is equivalent to the system of Eqs (1.6) or (1.7). It follows from this that system (1.6) (or (1.7)) is equivalent to Eq. (1.2) and is a certain non-trivial representation of it which resembles the Lax representation in ISPM [3]. In the case of steady inviscid flows, Eq. (1.3) with an arbitrary function $F(\phi)$ must be equivalent to the pair of equations

$$2\phi_{zz} = A_{zz}\phi, \quad 2\phi_{\bar z\bar z} = \overline{A}_{\bar z\bar z}\phi \tag{1.9}$$

containing a certain real function $A(z, \bar{z})$.

The equivalence of Eqs (1.6) to Eq. (1.2) and of Eqs (1.9) to Eq. (1.3) can also be established by another method which is based on the explicit calculation of the compatibility conditions for the pair of equations (1.9) and the pair of equations corresponding to it in (1.7). In the case of (1.9), this condition can be written in the form

$$\frac{\partial^2}{\partial z^2}\left(\frac{\phi_{zz}}{\phi}\right)-\frac{\partial^2}{\partial \bar{z}^2}\left(\frac{\phi_{\bar{z}\bar{z}}}{\phi}\right)=\frac{1}{4}i\left[D_1\left(\frac{1}{\phi}D_2\phi\right)-D_2\left(\frac{1}{\phi}D_1\phi\right)\right]=0$$

$$D_1=\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\quad D_2=\frac{\partial^2}{\partial x\partial y}$$

The existence of the following identity, which is satisfied in the case of any differentiable function $\psi = \ln\phi$ can be proved by direct verification

$$D_1\left(\frac{1}{\phi}D_2\phi\right)-D_2\left(\frac{1}{\phi}D_1\phi\right)\equiv\frac{\partial}{\partial y}\psi\frac{\partial}{\partial x}\Delta\psi-\frac{\partial}{\partial x}\psi\frac{\partial}{\partial y}\Delta\psi \tag{1.10}$$

which also provides a more elegant proof of the equivalence of (1.9) and (1.3). Using analogous calculations, one can transfer from system (1.7) or (1.6) to Eq. (1.2)

2. One of the simplest applications of the representation obtained is the possibility of a straightforward analysis of a number of properties of the initial system which is rather difficult to do starting out directly from its initial form (1.1) or the representation (1.2), (1.3).

1. The external friction can be eliminated from (1.6), and the system reduced to a simpler form. To do this, let us put

$$\phi=\phi'^{\alpha},\quad \alpha(t)=\alpha_0\exp\{\textstyle\int\kappa(t)dt\},\quad \nu'(t)=4\alpha(t)\nu$$

$$R=2\alpha R'-2\alpha(\alpha-1)\frac{\partial}{\partial z}\ln\phi',\quad t'=\int\frac{dt}{\alpha(t)} \tag{2.1}$$

System (1.6) then takes the form

$$\phi_t-\nu\phi\Delta\ln\phi=i(R_z-R_{\bar{z}})\phi/2$$

$$2\phi_{zz}=R_z\phi,\quad 2\phi_{\bar{z}\bar{z}}=\overline{R}_{\bar{z}}\phi$$

in which the prime on the variables has been omitted. It also follows from this that, when $\kappa\equiv 0$ the transformation (2.1) correspond to scale transformations of the variables so that, if $\phi(z, \bar{z}, t)$ is a solution of Eqs (1.6) the function $\phi^{\alpha}(z, \bar{z}, t)$ is also a solution of Eqs (1.6).

2. System (1.1) is invariant under a change to a reference system which is undergoing translational motion with an arbitrary variable velocity $\mathbf{C} = (u_0(t), \upsilon_0(t))$ with respect to the initial reference system. Here, the inertial forces are compensated by a redistribution of the pressure in the system. The corresponding transformations of the functions occurring in system (1.6) are as follows:

$$\phi\to\phi(z',\bar{z}',t')=\phi(z,\bar{z},t)\exp\{(zc+\bar{z}c)/2\}$$

$$R\to R'(z',\bar{z}',t')=R(z,\bar{z},t)-2c\ln\{\phi'\}+i(\bar{z}^2\dot{c}-c^2z)/2 \tag{2.2}$$

$$z\to z'=z+i\textstyle\int c(t)dt,\quad t'=t$$

$$c(t)=-\upsilon_0(t)-iu_0(t),\quad \dot{c}=dc/dt$$

3. System (1.1) is invariant under a change to a reference system which is uniformly rotating at a constant angular velocity ω_0 with respect to the initial reference system

$$\phi \to \phi(z',\bar{z}',t') = \phi(z,\bar{z},t)\exp\left\{\left(-\omega_0 z\bar{z} - 4\nu\omega_0 t\right)/2\right\}$$

$$R \to R'(z',\bar{z}',t') = \left[R(z,\bar{z},t) - 4\omega_0\bar{z}\ln\phi' - 2\bar{z}^2 z\omega_0\right]\exp\left\{-2i\omega_0 t\right\}$$

$$z \to z' = z\exp\left\{2i\omega_0 t\right\},\quad t' = t,\ \mathrm{Im}\left\{\omega_0\right\} = 0,\quad \omega_0 = \mathrm{const} \tag{2.3}$$

In this case, the inertial forces are again compensated by a redistribution of the pressure in the system.

To avoid mentioning the existence of time-dependent solutions associated with the above motions of the reference system as a whole, we shall refer to all the solutions as steady-state if a suitable reference system related to the initial one by transformations (2.2) and (2.3) is found such that $\phi'_{t'} = 0$, that is, $\phi(z', \bar{z}') = \phi(z, \bar{z}, t)$ in it, where z', $\bar{z}'$ are complex coordinates in the new reference system.

4. System (1.1) has a Cauchy–Lagrange integral which corresponds to a potential flow of the fluid

$$2\eta_t + u^2 + \upsilon^2 + E = \mathrm{const} \tag{2.4}$$

where η is the stream potential. The class of solutions which describes this type of flow within the framework of the present approach is parametrized as follows:

$$\phi(z,\bar{z},t) = \phi_0(z,t)\bar{\phi}_0(\bar{z},t) = |\phi_0(z,t)|^2$$

$$R(z,\bar{z},t) = r(z,t) + \bar{q}(\bar{z},t) \tag{2.5}$$

where the functions $r(z, t)$, $q(z, t)$ and $\phi_0(z, t)$ are related by the two equations

$$\phi_{0zz} = r_z\phi_0,\quad \phi_{0t} = q_z\phi_0 \tag{2.6}$$

so that one of the functions turns out to be arbitrary. In this case, the stream potential is equal to $\eta = i\ln\{\phi_0(z, t)/\bar{\phi}_0(\bar{z}, t)\}$. We also note the well-known fact that the solutions of Eqs (2.6) describe a purely vortex-free motion of a fluid if the function ϕ_0 is analytic in the whole of the complex plane. If, however, there should be poles, the corresponding solutions will describe potential flows with singular vortices.

The relation

$$u^2 + \upsilon^2 + E + \omega + \Delta A/2 = 0 \tag{2.7}$$

which follows from Eq. (1.8) is an analogue of the Cauchy–Lagrange integral (2.4) for fluid motion with a non-zero vorticity $\omega = \Delta\psi$. This exact relation, which plays the role of an energy integral, holds both in the case of an ideal fluid as well as in the case of a viscous fluid.

3. The representations obtained provide a natural method of constructing certain types of exact solutions of the initial equations. Actually, to analyse system (1.6), use can be made of the fact that the last two equations of this system are ordinary second-order differential equations. Hence, each of these equations taken separately has two linearly independent solutions. If one of them has the solutions ϕ_1 and ϕ_2, the second has the solutions $\bar{\phi}_1$ and $\bar{\phi}_2$ which are the complex conjugates of them. The requirement that just one of the linearly independent solutions of these equations should be real, that is, either $\phi_1 = \bar{\phi}_1$ or $\phi_2 = \bar{\phi}_2$ is a necessary condition for system (1.6) to be equivalent to the initial system (1.1). The simplest case corresponds to the reduction

$$\phi_1 = \bar{\phi}_1,\quad \phi_2 = \bar{\phi}_2 \tag{3.1}$$

It can be verified that, in the case of an ideal fluid, solutions of this type are steady-state solutions. Relations (3.1) imply the commutativity of the operators

$$L = \partial^2/\partial z^2 - R_z, \quad L = \partial^2/\partial \bar{z}^2 - \bar{R}_{\bar{z}}$$

Direct calculation of the commutator of these operators leads to the condition $R_z = f(z), \bar{R}_{\bar{z}} = \bar{f}(\bar{z})$ which, when the vorticity is non-zero, is equivalent to the condition for steady flow in accordance with the definition given above.

The simplest solution of this type can be constructed from the solutions of Eq. (2.6). Let $\phi_1(z)$ and $\phi_2(z)$ be two linearly independent solutions of Eq. (2.6) in the case of a certain function $R(z, \bar{z}) = r(z) + \bar{q}(\bar{z})$. In this case, the simplest steady-state solution which satisfies condition (3.1) will be the function

$$\Phi(z,\bar{z}) = \alpha|\phi_1(z)|^2 + \beta|\phi_2(z)|^2 + \gamma\phi_1(z)\bar{\phi}_2(\bar{z}) + \bar{\gamma}\phi_2(z)\bar{\phi}_1(\bar{z}) \tag{3.2}$$

where α and β are real, and γ is a complex constant. Let us now calculate the vorticity ω for the stream function $\psi = \ln\Phi(z, \bar{z})$: $\omega = \Delta\{\ln \Phi\}$. Substituting (3.2) into this formula, we obtain

$$\omega = \left(\alpha\beta - |\gamma|^2\right)|\sigma|^2\Phi^{-2} = \left(\alpha\beta - |\gamma|^2\right)|\sigma|^2 e^{-2\psi} \tag{3.3}$$

where $\sigma = \phi_1\phi'_2 - \phi_2\phi'_1 = \text{const}$. Equation (1.3) can therefore be written for this type of solution in the form of Liouville's equation

$$\Delta\psi = \left(\alpha\beta - |\gamma|^2\right)|\sigma|^2 e^{-2\psi}/4 \tag{3.4}$$

A simple generalization of (3.2), which again is a solution of Eq. (3.4), will be

$$\Phi(z,\bar{z}) = |\omega(z)|^2\left(\alpha|\phi_1(z)|^2 + \beta|\phi_2(z)|^2 + \gamma\phi_1(z)\bar{\phi}_2(\bar{z}) + \bar{\gamma}\phi_2(z)\bar{\phi}_1(\bar{z})\right) \tag{3.5}$$

where the functions ϕ_1 and ϕ_2 are linearly independent solutions of the equation

$$\phi_{zz} + (w_z w^{-1})\phi_z = r_z\phi \tag{3.6}$$

Although the solution (3.5) is obtained from (3.2) by a simple substitution for the function ϕ, it is nevertheless of some interest in view of the fact that, among the solutions (3.5), it is possible to pick out a particular class of solutions of the form

$$\begin{gathered} \Phi(z,z) = |w(z)|^2(N(s(z,\bar{z})) + C), \quad C = 0,+1,-1 \\ s(z,\bar{z}) = \int(w(z))^{-2}dz + \int(\bar{w}(\bar{z}))^{-2}d\bar{z} + s_0 \end{gathered} \tag{3.7}$$

In the case of an arbitrary function $w(z)$, we have $N(s) = \cos(\mu s)$ when $\mu = \text{const} \neq 0$, and $\text{Im}\,\{\mu\} = 0$ and $N(s) = s$ when $\mu = 0$. However, with a special choice of $w(z)$ in the case of an ideal fluid, two classes of solutions exist for which the structure of the streamlines is fixed, but the magnitude of the velocity in the streamlines can be arbitrary, that is, the function $N(s)$ is arbitrary and the functional dependence $s = s(z, \bar{z})$ is fixed. These are axially symmetric flows $\Phi(z, \bar{z}) = \Phi(r)$, $w(z) = z^{1/2}$ and solutions of the form $\Phi(z, \bar{z}) = \Phi(\chi)$, where $\chi = r^{-3/2}\cos\{\tfrac{3}{4}\theta + \theta_0\}$. Here, $r^2 = z\bar{z} = |z|^2$ and $\theta = (i/2)\ln\{z/\bar{z}\}$ is the polar angle. There is an analogous class of solutions in the case of a viscous fluid with a special choice of the function $w(z) = \lambda z^{1/2}$, where λ is an arbitrary complex constant, which is identical to the well-known class of generalized Hamel solutions [1] (see Section 5 of this paper).

We also note that, in the case of Liouville's equation, a general representation of the solutions is well known [2] which is referred to in [4, 5], for example, but not analysed in detail, and which can be written in the following form

$$\Phi(z,\bar{z}) = 4|\partial f(z)/\partial z|^2(1+|f(z)|^2)^{-2}$$

where $f(z)$ is an arbitrary analytic function. It can be shown that this solution is a special case of solutions of the type (3.7). The representations (3.2) and (3.5) which have been obtained are a more general form of this solution and they are more convenient when carrying out actual calculations.

4. The subclass of solutions which describe stationary waves close to a critical layer serves as an example of the use of the solutions (3.2) in applied problems. In Eq. (2.6), let us put $dr/dz = \xi^2$, $\xi = \text{const}$, $\operatorname{Im}\xi = 0$. Then, the functions $\phi_1(z) = \sin\xi z$, $\phi_2(z) = \cos\xi z$ may be chosen as the independent solutions $\phi_1(z)$ and $\phi_2(z)$. On substituting these functions into (3.2), we obtain

$$\Phi(z,\bar{z}) = \zeta(x,y) = a\,\mathrm{ch}\,\xi y + b\sin\xi x \tag{4.1}$$

Hence

$$u = \frac{\partial}{\partial y}\ln\zeta = \frac{\xi a\,\mathrm{sh}\,\xi y}{\zeta(x,y)},\quad \upsilon = -\frac{\partial}{\partial x}\ln\zeta = \frac{-\xi b\cos\xi x}{\zeta(x,y)} \tag{4.2}$$

The constants a and b therefore have the meaning of the wave amplitudes in the u and υ components of the flow. Let us now calculate the mean velocities

$$U(y) = \frac{\xi}{2\pi}\int_0^{2\pi/\xi} u(x,y)dx = a(a^2\,\mathrm{ch}^2\,\xi y - b^2)^{-1/2}\,\mathrm{sh}\,\xi y \tag{4.3}$$

$$V(y) = \frac{\xi}{2\pi}\int_0^{2\pi/\xi} \upsilon(x,y)dx = 0$$

When $a > b$, the flow under consideration is smooth and represents a wave in a plane-parallel shear flow with a mean velocity $U(y)$. The wave amplitude maximum occurs at the critical point $y = 0$ where the phase velocity of the wave is identical to the flow velocity. This solution was obtained in the case of a reference system moving at the phase velocity of the wave.

When $a = b$, a periodic change of singular vortices, which move at the phase velocity of the wave, appears along the line of the critical layer. In this case, the mean velocity is constant throughout the whole fluid layer and undergoes a finite discontinuity at the point $y = 0$: $U(y) = \operatorname{sign}(y)$. This flow regime is potential: $\omega = \Delta\psi = 0$.

When $b > a$, the singular vortices are converted into finite vortex lacunae which move at the phase velocity of the wave, on the edges of which the velocity becomes infinite. Outside of the domain $a^2\,\mathrm{ch}\,\xi y < b^2$, the mean velocity is described by the same formula (4.3). Within the domain, the mean velocity is undefined.

Using the technique for the constructing multi-soliton solutions [3], it is possible to construct a large number of exact solutions which describe a flow when there are several critical layers with a constant average velocity at infinity. Similar solutions can also be written out in polar coordinates.

5. In conclusion, let us consider two types of viscous flows which are closely associated with the preceding analysis of exact solutions in the case of ideal fluid flows. Solutions, which correspond to representation (3.7), are these special classes of solutions. With the special choice of $w(z) = \lambda z^{1/2}$, where λ is a real constant, the equation for $N(s)$ takes the form of the following equation

$$\partial\ln N/\partial t - \nu e^{-2s}\partial^2\ln N/\partial s^2 = 0 \tag{5.1}$$

which describes unsteady, axially symmetric flows of a viscous fluid. In the case when λ is a complex constant, the solutions reduce to the well-known steady-state Hamel solutions.

Let $\lambda - k + im$. The complete system of equations which describes flows of this type will then have the form

$$\frac{d^2}{ds^2}\left(\nu \ln N - \frac{1}{2}B\right) = 0 \tag{5.2}$$

$$2\frac{1}{N}\frac{d^2N}{ds^2} = \frac{d^2A}{ds^2} + i\frac{d^2B}{ds^2} - (k+im)\frac{d}{ds}(A+iB)$$

Hence, eliminating A and B, we arrive at a single equation in $\chi(s) = dN/ds$

$$\chi'' - (2k + \frac{m}{2\nu})\chi' + (m^2+k^2)\chi - \frac{m}{2\nu}\chi^2 + \alpha(m^2+k^2) = 0 \tag{5.3}$$

(a prime denotes differentiation with respect to s). In this case

$$s = \ln\left|z^{1/\lambda}\right| = \frac{2k}{m^2+k^2}\ln r + \frac{2m}{m^2+k^2}\theta$$

Here $r = |z|$ and $\theta = (i/2)\ln(z/\bar{z})$ are the radial and angular coordinates, respectively. Equations (5.3) precisely correspond to the equations of generalized Hamel flows [1].

6. The above calculations show that the representation (1.6)–(1.9) of the equations of the dynamics of an incompressible fluid can be useful when investigating flow structure and is possibly richer in content than the representation (1.2), (1.3). Representations which are analogous to (1.6) can be written out for the equations of fluid dynamics in the case of more general conditions such as, for example, the case of a compressible ideal fluid or two-dimensional flows on a sphere which is of interest in geophysics.

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